

Differential equations with discrete state-dependent delay: uniqueness and well-posedness in the space of continuous functions

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Abstract. Partial differential equations with discrete (concentrated) state-dependent delays in the space of continuous functions are investigated. In general, the corresponding initial value problem is not well posed, so we find an additional assumption on the state-dependent delay function to guarantee the well posedness. For the constructed dynamical system we study the long-time asymptotic behavior and prove the existence of a compact global attractor.

Key words : Partial functional differential equation, state-dependent delay, delay selection, well-posedness, global attractor

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1. Introduction

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Delay differential equations is one of the oldest branches of the theory of infinite dimensional dynamical systems - theory which describes qualitative properties of systems, changing in time.

We refer to the classical monographs on the theory of ordinary (O.D.E.) delay equations [Hale (1977), Hale and Lunel (1993), Diekmann et al. (1995), Azbelev et al. (1991), Mishkis (1972)]. The theory of partial (P.D.E.) delay equations is essentially less developed since such equations are infinite-dimensional in both time (as delay equations) and space (as P.D.E.s) variables, which makes the analysis more difficult. We refer to some works which are close to the present research [Travis and Webb (1974), Chueshov (1992), Chueshov and Rezounenko (1995), Boutet de Monvel et al. (1998), Rezounenko (2003)] and to the monograph [Wu (1996)].

Recently, a new class of delay equations - equations with a state-dependent delay (SDD) attracts much attention of researchers (see e.g. [Walther (2002), Walther (2003), Nussbaum and Mallet-Paret (1992), Nussbaum and Mallet-Paret (1996), Mallet-Paret et al. (1994), Krisztin (2003), Walther (2007)] and also the review [Hartung et al (2006)] for details and references). Investigations of these equations essentially differ from the ones of equations with constant or time-dependent delays. The main difficulty is that nonlinearities with SDD (in contrast to constant or time-dependent delays) are not Lipschitz continuous on the space of continuous functions - the main space of initial data, where the classical theory of delay equations is developed (see the references above). As a consequence, the corresponding initial value problem (IVP), in general, is **not** well posed (in the sense of J. Hadamard [Hadamard (1902), Hadamard (1932)]). An explicit example

of the nonuniqueness of solutions to an ordinary equation with state-dependent delay (SDD) is given in [Winston (1970)] (see also [Hartung et al (2006), p.465]). As noticed in [Hartung et al (2006), p.465] "typically, the IVP is uniquely solved for initial and other data which satisfy suitable Lipschitz conditions." The idea to investigate SDD O.D.E.s in the space of Lipschitz continuous functions is very fruitful (see the references above).

Unfortunately, in contrast to O.D.E.s with state-dependent delay, one has **no** possibility to exploit the space of Lipschitz continuous functions for P.D.E.s with SD delay, because solutions of P.D.E.s usually do not belong to this space.

The first attempt to treat P.D.E.s with state-dependent (state-selective) delays has been made for a distributed delay problem [Rezounenko and Wu (2006), Rezounenko (2007)]. The existence (without uniqueness) of solutions for P.D.E.s with discrete state-dependent delay was studied in [Hernandez et al (2006)] (mild solutions) and [Rezounenko (2007)] (weak solutions).

As noticed above, it is a common point of view [Hartung et al (2006)] that equations (O.D.E.s and P.D.E.s) with discrete state-dependent delay are **not** well posed in the space of continuous functions (C). This leads to the search of (particular) classes of equations which may be well-posed in the space of continuous functions (C).

The main goal of the present paper is to propose an assumption on the delay, which is sufficient for the well-posedness of the corresponding initial value problem in the space C. To the best of our knowledge, this is the first result for the well-posedness in C of the discrete state-dependent delay (for P.D.E.s as well as for O.D.E.s).

Discussing the meaning of the main assumption (H) (see below) for applied problems, we hope that the assumption is the natural mathematical expression of the fact that for many applied problems, the models have a parameter (time $\eta_{ign} > 0$) which is necessary to take into considerations the time changes in the system. The changes not always can be taken into considerations immediately. To this end, the existence of $\eta_{ign} > 0$ (no matter how small the value of $\eta_{ign} > 0$ is!) makes the corresponding initial value problem well posed.

2. Formulation of the model with state-dependent discrete delay

Our goal is to present an approach to study the following partial differential equation with *state-dependent discrete delay*

$$\frac{\partial}{\partial t}u(t, x) + Au(t, x) + du(t, x) = \int_{\Omega} b(u(t - \eta(u_t), y))f(x - y)dy \equiv (F(u_t))(x), \quad x \in \Omega, \quad (1)$$

where A is a densely-defined self-adjoint positive linear operator with domain $D(A) \subset L^2(\Omega)$ and with compact resolvent, so $A : D(A) \rightarrow L^2(\Omega)$ generates an analytic semi-group, Ω is a smooth bounded domain in R^{n_0} , $f : \Omega - \Omega \rightarrow R$ is a bounded function to be specified later, $b : R \rightarrow R$ is a locally Lipschitz map, d is a non-negative constant. The function $\eta(\cdot) : C([-r, 0]; L^2(\Omega)) \rightarrow R_+$ represents the state-dependent *discrete* delay. We denote for short $C \equiv C([-r, 0]; L^2(\Omega))$. The norms in $L^2(\Omega)$ and C are denoted by $\|\cdot\|$ and $\|\cdot\|_C$ respectively. As usually for delay equations, we denote by u_t the function of $\theta \in [-r, 0]$ by the formula $u_t \equiv u_t(\theta) \equiv u(t + \theta)$.

We consider equation (1) with the following initial condition

$$u|_{[-r,0]} = \varphi \in C \equiv C([-r, 0]; L^2(\Omega)). \quad (2)$$

The methods used in our work can be applied to another types of nonlinear and delay P.D.E.s (as well as O.D.E.s). We choose a particular form of nonlinear delay terms F for simplicity and to illustrate our approach on the diffusive Nicholson's blowflies equation (see the end of the article for more details). Below you will also find a remark (Rem. 11) on the local in space variable problems.

3. The existence of mild solutions

In our study we use the standard

Definition 1. *A function $u \in C([-r, T]; L^2(\Omega))$ is called a mild solution on $[-r, T]$ of the initial value problem (1), (2) if it satisfies (2) and*

$$u(t) = e^{-At}\varphi(0) + \int_0^t e^{-A(t-s)} \{F(u_s) - d \cdot u(s)\} ds, \quad t \in [0, T]. \quad (3)$$

Proposition 1. *Assume the function $b : R \rightarrow R$ is a locally Lipschitz map, satisfying $|b(w)| \leq C_1|w| + C_b$ with $C_i \geq 0$, and function $\eta(\cdot) : C([-r, 0]; L^2(\Omega)) \rightarrow R_+$ is continuous, $f : \Omega - \Omega \rightarrow R$ is a bounded function. Then for any initial function $\varphi \in C$, initial value problem (1), (2) has a global mild solution which satisfies $u \in C([-r, +\infty); L^2(\Omega))$.*

The existence of a mild solution is a consequence of the continuity of $F(\varphi) \equiv \int_{\Omega} b(\varphi(-\eta(\varphi), y))f(\cdot - y)dy : C \rightarrow L^2(\Omega)$ (see (1)) which gives the possibility to use the standard method based on Schauder fixed point theorem (see e.g. [Wu (1996), theorem 2.1, p.46]). The solution is also global (is defined for all $t \geq -r$) see e.g. [Wu (1996), theorem 2.3, p.49].

Remark 1. It is important to notice that even in the case of ordinary differential equations (even scalar) the mapping of the form $\tilde{F}(\varphi) = f(\varphi(-r(\varphi))) : C([-r_0, 0]; R) \rightarrow R$ has a very unpleasant property. The authors in [Louihi et al (2002), p.3] write "Notice that the functional \tilde{F} is defined on C , but it is clear that it is neither differentiable nor locally Lipschitz continuous, whatever the smoothness of f and r ." As a consequence, the Cauchy problem associated with equations with such a nonlinearity "...is not well posed in the space of continuous functions, due to the non-uniqueness of solutions whatever the regularity of the functions f and r " [Louihi et al (2002), p.2]. See also a detailed discussion in [Hartung et al (2006)].

4. Uniqueness and well-posedness

As in the previous section, we assume that $\eta : C \rightarrow R_+$ is continuous and $f : \Omega - \Omega \rightarrow R$ is a bounded function ($|f(\cdot)| \leq M_f$). Additionally, we assume the following assumption on the delay function η is satisfied

- $\exists \eta_{ign} > 0$ such that η "ignores" values of $\varphi(\theta)$ for $\theta \in (-\eta_{ign}, 0]$ i.e.

$$\exists \eta_{ign} > 0 : \forall \varphi^1, \varphi^2 \in C : \forall \theta \in [-r, -\eta_{ign}], \Rightarrow \varphi^1(\theta) = \varphi^2(\theta) \implies \eta(\varphi^1) = \eta(\varphi^2). \quad (H)$$

Remark 2. It is easy to present many examples of (delay) functions η , which satisfy assumption (H). Some of them are

$$\begin{aligned}\eta(\varphi) &= p_1(\varphi(-r)) \text{ with } p_1 : L^2(\Omega) \rightarrow R_+; \\ \eta(\varphi) &= \sum_{k=1}^N p_k(\varphi(-r_k)) \text{ with } p_k : L^2(\Omega) \rightarrow R_+; \quad \min r_k > 0; \\ \eta(\varphi) &= \int_{-r}^{-\eta_{ign}} p_1(\varphi(\theta)) d\theta, \quad \text{and} \quad \eta(\varphi) = p_1 \left(\int_{-r}^{-\eta_{ign}} \varphi(\theta) d\theta \right), \quad \eta_{ign} > 0, \text{ etc.}\end{aligned}$$

Remark 3. It is interesting to notice that in the case $\eta_{ign} > r$, we have that the delay function η ignores all values of $\varphi(\theta)$, $\forall \theta \in [-r, 0]$, so $\eta(\varphi) \equiv \text{const}$, $\forall \varphi \in C$ i.e. equation (1) becomes equation with constant delay. On the other hand, assumption, similar to (H) with $\eta_{ign} = 0$, is trivial since $\varphi^1(\theta) = \varphi^2(\theta)$ for all $\theta \in [-r, 0]$ means $\varphi^1 = \varphi^2$ in C , so $\eta(\varphi^1) = \eta(\varphi^2)$.

To show that assumption (H) implies the *uniqueness* of mild solutions (given by Proposition 1), we will use the standard method of steps with a step less than $\eta_{ign} > 0$. First, let us introduce, for any $\varphi \in C$ the extension function

$$\overline{\varphi}(s) \equiv \begin{cases} \varphi(s) & s \in [-r, 0]; \\ \varphi(0) & s \in (0, \eta_{ign}) \end{cases}.$$

Consider any mild solution $u(t)$ of IVP (1), (2) and the nonlinearity $\int_{\Omega} b(u(t - \eta(u_t, y)))f(\cdot - y)dy$. For all $t \in [0, \eta_{ign})$ assumption (H) gives $\eta(u_t) = \eta(\overline{\varphi}_t)$. Let us denote by $r^\varphi(t) \equiv \eta(\overline{\varphi}_t)$, $t \in [0, \eta_{ign})$.

Hence any mild solution $u(t)$ of IVP (1), (2) for all values of $t \in [0, \eta_{ign})$ is also a solution of

$$\begin{cases} \dot{u}(t) + Au(t) + d \cdot u(t) = \int_{\Omega} b(u(t - r^\varphi(t), y))f(\cdot - y)dy, & t \in [0, \eta_{ign}), \\ u(\theta) = \varphi(\theta), \theta \in [-r, 0], \end{cases} \quad (4)$$

where $r^\varphi(t)$ is *time*-dependent (but not *state*-dependent delay) known function for all $t \in [0, \eta_{ign})$. To show that the last Cauchy problem (with time-dependent delay) has the unique solution, it is sufficient to consider any two solutions of (4) and their difference $w(t) \equiv u^1(t) - u^2(t)$, which satisfies (c.f. (3))

$$\begin{aligned}w(t) &= \int_0^t e^{-A(t-s)} \times \\ &\left\{ \int_{\Omega} [b(u^1(s - r^\varphi(s), y)) - b(u^2(s - r^\varphi(s), y))] f(\cdot - y)dy - d \cdot w(s) \right\} ds, \quad t \in [0, \eta_{ign}).\end{aligned} \quad (5)$$

An easy calculation, the local Lipschitz property of b and $\|e^{-A(t-s)}\| \leq 1$ give

$$\|w(t)\| \leq \int_0^t (M_f |\Omega| L_b + d) \max_{s \in [0, t]} \|w(s)\| ds \leq t \cdot (M_f |\Omega| L_b + d) \max_{s \in [0, t]} \|w(s)\|.$$

Here we denote $|\Omega| \equiv \int_{\Omega} 1 \cdot dx$.

Remark 4. Here we used that $\max_{s \in [-r, t]} \|w(s)\| = \max_{s \in [0, t]} \|w(s)\|$ since $w(s) \equiv 0$ for $\theta \in [-r, 0]$ (w is the difference of two solutions, both satisfying (2)).

Choose small enough $\alpha > 0$ to satisfy $\alpha(M_f|\Omega|L_b + d) < 1$. The last estimate gives

$$\max_{s \in [0, \alpha]} \|w(s)\| \leq \alpha \cdot (M_f|\Omega|L_b + d) \max_{s \in [0, \alpha]} \|w(s)\| < \max_{s \in [0, \alpha]} \|w(s)\|$$

which implies $\max_{s \in [0, \alpha]} \|w(s)\| = 0$. This means that any two mild solutions of (4) coincide for $t \in [0, \alpha]$ with $\alpha < (M_f|\Omega|L_b + d)^{-1}$. Repeat this considerations (if necessary) by steps of length α to cover $[0, \eta_{ign})$. This gives the uniqueness of solutions of (4) and hence the uniqueness of solutions of (1), (2) for $t \in [0, \eta_{ign})$. The uniqueness on any interval $[0, T]$ holds by the similar arguments on each time interval $[k \cdot \eta_{ign}, (k + 1) \cdot \eta_{ign})$, $k \in N$ (redefining the function $r^\varphi(t)$).

We may define the **evolution operator** $S_t : C \rightarrow C$ by the formula $S_t \varphi \equiv u_t$, where $u(t)$ is the unique mild solution of (1), (2) with initial function φ .

Remark 5. The system becomes much simpler if we additionally assume that the delay function η satisfies

$$\exists \eta_{min} > 0 \text{ such that } \eta : C \rightarrow [\eta_{min}, r] \text{ is continuous.} \quad (H1)$$

In that case we may use the classical method of steps with a step less than $\min\{\eta_{ign}, \eta_{min}\}$. To satisfy assumption (H1) for the functions given in Remark 2 it is sufficient to assume that $\inf p_i(\cdot) > 0$.

Remark 6. For applied problems described by ordinary differential equations, condition $\eta(\cdot) \in [\eta_{min}, r]$ (see (H1)) is used and motivated in [Al-Omari, Gourley (2005), p.15] and [Aiello, Freedman and Wu (1992)]. The authors write "This assumption is known to be realistic in the case of Antarctic whale and seal populations."

Remark 7. It is interesting to notice that without (H1), one cannot say that the nonlinearity $F(u_t)$ depends on $u|_{[-r, -\eta_{ign}]}$ only. The SD-delay may vanish (i.e. (H1) does not hold).

The main result of this section is the following

Theorem 1. Assume the function $b : R \rightarrow R$ is a locally Lipschitz map, satisfying $|b(w)| \leq C_1|w| + C_b$ with $C_i \geq 0$, the delay function $\eta : C \rightarrow R_+$ is continuous and satisfies the assumption (H), $f : \Omega - \Omega \rightarrow R$ is a bounded function ($|f(z)| \leq M_f, \forall z \in \Omega - \Omega$). Then the pair (S_t, C) constitutes a dynamical system i.e. the following properties are satisfied:

1. $S_0 = Id$ (identity operator in C);
2. $\forall t, \tau \geq 0 \implies S_t S_\tau = S_{t+\tau}$;
3. $t \mapsto S_t$ is a strongly continuous in C mapping;
4. for any $t \geq 0$ the evolution operator S_t is continuous in C i.e. for any $\{\varphi^n\}_{n=1}^\infty \subset C$ such that $\|\varphi^n - \varphi\|_C \rightarrow 0$ as $n \rightarrow \infty$, one has $\|S_t \varphi^n - S_t \varphi\|_C \rightarrow 0$ as $n \rightarrow \infty$.

Remark 8. *Theorem 1 particularly means that the initial value problem (1), (2) is well posed in the space C in the sense of J. Hadamard [Hadamard (1902), Hadamard (1932)].*

Remark 9. *It is important to emphasize, that we do not assume the SD-delay function η to be Lipschitz (c.f. [Rezounenko (2007)]). We propose an alternative approach, based on the assumption (H) which is of different nature to the Lipschitz property of η .*

Proof. Properties 1), 2) are consequences of the uniqueness of mild solutions due to (H) (see considerations above). Property 3) is given by Proposition 1 since the solution is a continuous function $u \in C([-r, T]; L^2(\Omega))$.

Let us prove property 4. Let us fix $t_0 \in [0, \eta_{ign})$. Denote by $u^k(t)$ the solution of (1),(2) with the initial function φ^k and by $u(t)$ the solution of (1),(2) with the initial function φ .

We use the variation of constants formula for parabolic equation (4) (with $\tilde{A} \equiv A + d \cdot$)

$$u(t) = e^{-\tilde{A}t}u(0) + \int_0^t e^{-\tilde{A}(t-\tau)} \int_{\Omega} b(u(\tau - \eta(u_{\tau}), y)) f(\cdot - y) dy d\tau, \quad (6)$$

$$u^k(t) = e^{-\tilde{A}t}u^k(0) + \int_0^t e^{-\tilde{A}(t-\tau)} \int_{\Omega} b(u^k(\tau - \eta(u_{\tau}^k), y)) f(\cdot - y) dy d\tau. \quad (7)$$

Using $\|e^{-\tilde{A}t}\| \leq 1$ and $\|e^{-\tilde{A}(t-\tau)}\| \leq 1$, we get

$$\begin{aligned} \|u^k(t) - u(t)\| &\leq \|u^k(0) - u(0)\| \\ &+ \int_0^t \left\| \int_{\Omega} [b(u^k(\tau - \eta(u_{\tau}^k), y)) - b(u(\tau - \eta(u_{\tau}), y))] f(\cdot - y) dy \right\| d\tau \\ &= \|\varphi^k(0) - \varphi(0)\| + \mathcal{J}_1^k(t) + \mathcal{J}_2^k(t), \end{aligned} \quad (8)$$

where we denote

$$\mathcal{J}_1^k(s) \equiv \int_0^s \left\| \int_{\Omega} [b(u^k(\tau - \eta(u_{\tau}^k), y)) - b(u(\tau - \eta(u_{\tau}), y))] f(\cdot - y) dy \right\| d\tau, \quad (9)$$

$$\mathcal{J}_2^k(s) \equiv \int_0^s \left\| \int_{\Omega} [b(u(\tau - \eta(u_{\tau}^k), y)) - b(u(\tau - \eta(u_{\tau}), y))] f(\cdot - y) dy \right\| d\tau. \quad (10)$$

Using the Lipschitz property of b , one easily gets

$$\mathcal{J}_1^k(t) \leq M_f |\Omega| L_b \int_0^t \|u^k(\tau - \eta(u_{\tau}^k)) - u(\tau - \eta(u_{\tau}))\| d\tau \leq M_f |\Omega| L_b t \max_{s \in [-r, t]} \|u^k(s) - u(s)\|. \quad (11)$$

Estimates (11), (8) and property $\mathcal{J}_2^k(s) \leq \mathcal{J}_2^k(t)$ for $s \leq t \leq t_0 < \eta_{ign}$ give

$$\max_{t \in [0, t_0]} \|u^k(t) - u(t)\| \leq \|\varphi^k(0) - \varphi(0)\| + M_f |\Omega| L_b t_0 \max_{s \in [-r, t_0]} \|u^k(s) - u(s)\| + \mathcal{J}_2^k(t_0).$$

Hence

$$\max_{s \in [-r, t_0]} \|u^k(s) - u(s)\| \leq \|\varphi^k - \varphi\|_C + M_f |\Omega| L_b t_0 \max_{s \in [-r, t_0]} \|u^k(s) - u(s)\| + \mathcal{J}_2^k(t_0).$$

We choose $t_0 < [M_f|\Omega|L_b]^{-1}$ (to satisfy $M_f|\Omega|L_bt_0 < 1$) and get

$$(1 - M_f|\Omega|L_bt_0) \max_{s \in [-r, t_0]} \|u^k(s) - u(s)\| \leq \|\varphi^k - \varphi\|_C + \mathcal{J}_2^k(t_0). \quad (12)$$

Our goal is to show that $\mathcal{J}_2^k(t_0) \rightarrow 0$ as $k \rightarrow \infty$. The Lipschitz property of b implies

$$\mathcal{J}_2^k(t_0) \leq M_f|\Omega|L_b \int_0^{t_0} \|u(\tau - \eta(u_\tau^k)) - u(\tau - \eta(u_\tau))\| d\tau. \quad (13)$$

We use the extension functions

$$\overline{\varphi}(s) \equiv \begin{cases} \varphi(s) & s \in [-r, 0]; \\ \varphi(0) & s \in (0, \eta_{ign}) \end{cases} \quad \text{and} \quad \overline{\varphi}^k(s) \equiv \begin{cases} \varphi^k(s) & s \in [-r, 0]; \\ \varphi^k(0) & s \in (0, \eta_{ign}) \end{cases}.$$

It is easy to see that the convergence $\|\varphi^k - \varphi\|_C \rightarrow 0$ implies $\|\overline{\varphi}_\tau^k - \overline{\varphi}_\tau\|_C \rightarrow 0$ for any $\tau \in [0, \eta_{ign})$.

On the other hand, for any $\tau \in [0, \eta_{ign}]$ and any $\theta \in [-r, -\eta_{ign}]$ we have $u_\tau^k(\theta) = \varphi^k(\tau + \theta)$, hence assumption (H) gives $\eta(u_\tau^k) = \eta(\overline{\varphi}_\tau^k)$ for any $\tau \in [0, \eta_{ign})$. The same arguments give $\eta(u_\tau) = \eta(\overline{\varphi}_\tau)$ for any $\tau \in [0, \eta_{ign})$.

The considerations above show that the convergence $\|\varphi^k - \varphi\|_C \rightarrow 0$ implies $|\eta(u_\tau^k) - \eta(u_\tau)| = |\eta(\overline{\varphi}_\tau^k) - \eta(\overline{\varphi}_\tau)| \rightarrow 0$ for all $\tau \in [0, \eta_{ign})$. Here we used the continuity of $\eta : C \rightarrow R_+$.

The last property gives that for all $\tau \in [0, \eta_{ign})$ one has $\tau - \eta(u_\tau^k) \rightarrow \tau - \eta(u_\tau)$, when $k \rightarrow \infty$. Hence the continuity of the mild solution u (the strong continuity in $L^2(\Omega)$) implies (see the integral in (13))

$$\forall \tau \in [0, t_0] \implies \|u(\tau - \eta(u_\tau^k)) - u(\tau - \eta(u_\tau))\| \rightarrow 0, \quad \text{when } k \rightarrow \infty. \quad (14)$$

On the other hand, it is evidently that

$$\forall \tau \in [0, t_0] \implies \|u(\tau - \eta(u_\tau^k)) - u(\tau - \eta(u_\tau))\| \leq 2 \max_{s \in [-r, t_0]} \|u(s)\| < +\infty. \quad (15)$$

Properties (14) and (15) allow us to use Lebesgue-Fatou lemma ([Yosida (1965), p.32]) for the scalar function $\|u(\tau - \eta(u_\tau^k)) - u(\tau - \eta(u_\tau))\|$ to conclude that

$$\int_0^{t_0} \|u(\tau - \eta(u_\tau^k)) - u(\tau - \eta(u_\tau))\| d\tau \rightarrow 0 \quad \text{when } k \rightarrow \infty. \quad (16)$$

Estimates (16) and (13) prove that $\mathcal{J}_2^k(t_0) \rightarrow 0$ as $k \rightarrow \infty$.

Since

$$\max_{t \in [0, t_0]} \mathcal{J}_2^k(t) \leq \mathcal{J}_2^k(t_0) \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

we finally conclude (see (12) and the last properties) that for all $t \in [0, t_0]$:

$$\begin{aligned} \|u_t^k - u_t\|_C &\equiv \max_{\theta \in [-r, 0]} \|u^k(t + \theta) - u(t + \theta)\|_{L^2(\Omega)} \\ &\leq [1 - M_f|\Omega|L_bt_0]^{-1} \cdot (\|\varphi^k - \varphi\|_C + \mathcal{J}_2^k(t_0)) \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned} \quad (17)$$

Estimate (17) gives that $\|u_t^k - u_t\|_C \rightarrow 0$ when $\|\varphi^k - \varphi\|_C \rightarrow 0$ for all $t \in [0, t_0] \subset [0, \min\{[M_f|\Omega|L_b]^{-1}, \eta_{ign}\})$. This is the strong continuity in the space C of the evolution operator S_t for all (small) $t \in [0, t_0] \subset [0, \min\{[M_f|\Omega|L_b]^{-1}, \eta_{ign}\})$.

For any $t \geq 0$ we present $S_t\varphi$ as the composition of mappings

$$S_t\varphi = \underbrace{S_p \circ S_p \circ S_p \circ \dots \circ S_p}_{q \text{ times}} \circ S_{t-[t \cdot (2p)^{-1}]} \varphi, \quad \text{where} \quad p \equiv \frac{1}{2} \min\{L_b^{-1}, \eta_{ign}\}, \quad q \equiv \left\lceil \frac{t}{2p} \right\rceil. \quad (18)$$

Here $[\cdot]$ denotes the integer part of a real number. The continuity of S_t follows from the proved continuity of S_p and $S_{t-[t \cdot (2p)^{-1}]}$ (since both p and $t - [t \cdot (2p)^{-1}]$ belong to $[0, \min\{[M_f|\Omega|L_b]^{-1}, \eta_{ign}\})$). The property 4 is proved. It complies the proof of Theorem 1. \blacksquare

5. Asymptotic behavior

This section is devoted to the study of the long-time asymptotic behavior of the dynamical system (S_t, C) , constructed in Theorem 1.

Theorem 2. *Assume the function $b : R \rightarrow R$ is a locally Lipschitz bounded map ($|b(w)| \leq C_b$ with $C_b \geq 0$) and the delay function $\eta : C \rightarrow R_+$ is continuous and satisfies the assumption (H), $f : \Omega - \Omega \rightarrow R$ is a bounded function ($|f(\cdot)| \leq M_f$). Then the dynamical system (S_t, C) has a compact global attractor which is a compact set in all spaces $C_\delta \equiv C([-r, 0]; D(A^\delta))$, $\forall \delta \in [0, \frac{1}{2})$.*

Proof. Our proof is split on four steps.

Step 1. Let us first prove that for any $T > 0$ and any bounded set $B \subset C$ there exists a constant $C_T(B)$ such that for any mild solution of (1), (2) with initial values in B , one has

$$\forall T > 0, \forall B, \exists C_T(B) > 0 : \forall t \in [0, T] \implies \|u(t)\| \leq C_T(B). \quad (19)$$

Equation (3) implies $\|u(t)\| \leq \|\varphi(0)\| + \int_0^t (\|F(u_s)\| + d\|u(s)\|) ds$. Using $\|F(u_s)\| \leq M_f|\Omega|^{3/2}C_b$, we have $\|u(t)\| \leq \|\varphi(0)\| + d \int_0^t \|u(s)\| ds + tM_f|\Omega|^{3/2}C_b$. Denote by $\Psi(t) \equiv \int_0^t \|u(s)\| ds$ and use Gronwall lemma to get $\Psi(t) \leq \|\varphi(0)\|d^{-1}e^{dt} + e^{dt}M_f|\Omega|^{3/2}C_b d^{-2}[1 - e^{-dt}(dt - 1)]$. The last estimate gives

$$\|u(t)\| \leq \|\varphi(0)\| + d \cdot \left[\|\varphi(0)\|d^{-1}e^{dt} + e^{dt}M_f|\Omega|^{3/2}C_b d^{-2}[1 - e^{-dt}(dt - 1)] \right] + tM_f|\Omega|^{3/2}C_b.$$

It implies (19).

Step 2. Next we show that a solution becomes more smooth for positive t .

Lemma. *For any $0 < \alpha < 1$, $\epsilon > 0$, $T > 0$ and any bounded set $B \subset C$ there exists a constant $C_{\alpha, \epsilon, T}(B)$ such that for any mild solution of (1), (2) with initial values in B , one has*

$$\|A^\alpha u(t)\| \leq C_{\alpha, \epsilon, T}(B) \quad \text{for } t \in (\epsilon, T]. \quad (20)$$

The proof of the lemma is standard (see e.g. [Chueshov (1999)] and also Rezounenko-MAG-1997, Rezounenko-Wu-2006).

Step 3. Dissipativeness. Lemma implies that $u(t) \in D(A^{\frac{1}{2}+\delta}), \forall \delta \in [0, 1/2), t \geq \varepsilon(\delta)$. Hence $Au(t) \in D(A^{-\frac{1}{2}+\delta}), \forall \delta \in [0, 1/2), t \geq \varepsilon(\delta)$. Equation (1) gives $\dot{u}(t) \in D(A^{-\frac{1}{2}+\delta}), \forall \delta \in [0, 1/2), t \geq \varepsilon(\delta)$. So for all $t \geq \varepsilon(\delta)$ one has

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|A^\delta u(t)\|^2 + \|A^{\frac{1}{2}+\delta} u(t)\|^2 + d \cdot \|A^\delta u(t)\|^2 &\leq \|F(u_t)\| \cdot \|A^{2\delta} u(t)\| \\ &\leq M_f |\Omega|^{3/2} C_b \cdot \lambda_1^{2\delta-1} \|A^{\frac{1}{2}+\delta} u(t)\| \leq \frac{1}{2} M_f^2 |\Omega|^3 C_b^2 \lambda_1^{4\delta-2} + \frac{1}{2} \|A^{\frac{1}{2}+\delta} u(t)\|^2. \end{aligned}$$

Hence $\frac{d}{dt} \|A^\delta u(t)\|^2 + \|A^{\frac{1}{2}+\delta} u(t)\|^2 + 2d \cdot \|A^\delta u(t)\|^2 \leq M_f^2 |\Omega|^3 C_b^2 \lambda_1^{4\delta-2}$. Using $\|A^{\frac{1}{2}+\delta} v\|^2 \geq \lambda_1 \|A^\delta v\|^2$ we have $\frac{d}{dt} \|A^\delta u(t)\|^2 + (\lambda_1 + 2d) \|A^\delta u(t)\|^2 \leq M_f^2 |\Omega|^3 C_b^2 \lambda_1^{4\delta-2}$. Gronwall's lemma gives $\|A^\delta u(t)\|^2 \leq \|A^\delta u(\varepsilon)\|^2 \cdot \exp\{-(\lambda_1 + 2d)t\} + M_f^2 |\Omega|^3 C_b^2 \lambda_1^{4\delta-2} (\lambda_1 + 2d)^{-1}$. By lemma (step 2), the value $\|A^\delta u(\varepsilon)\|$ is finite, which implies that

$$\begin{aligned} \forall \delta \in [0, 1/2) \quad \exists C(\delta) > 0 : \forall B - \text{bounded in } C, \exists t(B, \delta) : \\ \forall t \geq t(B, \delta) \implies \|A^\delta u(t)\| \leq C(\delta). \end{aligned} \quad (21)$$

Step 4. Our next step is to show that the set $\{S_t \varphi \mid \varphi \in B, t > r + \varepsilon\}$ is an equicontinuous family in $C_\delta \equiv C([-r, 0]; D(A^\delta)), \forall \delta \in [0, \frac{1}{2})$.

Remark 10. *In our case we cannot use [Wu (1996), thm. 1.8, p.42] since our nonlinearity F is not Lipschitz.*

We denote by $\{e_k\}_{k=1}^\infty$ the orthonormal basis (of $L^2(\Omega)$) of eigenvectors of the operator A , so $Ae_k = \lambda_k e_k$, $0 < \lambda_1 < \lambda_2 < \dots < \lambda_k \rightarrow +\infty$. Consider for $v \in L^2(\Omega)$

$$\|A^\delta (e^{-At_1} v - e^{-At_2} v)\|^2 = \sum_{k=1}^\infty (e^{-\lambda_k t_1} - e^{-\lambda_k t_2})^2 \lambda_k^{2\delta} \cdot v_k^2, \quad \text{where } v_k \equiv \langle v, e_k \rangle. \quad (22)$$

Assuming $0 < t_1 < t_2$, one can easily check that $(e^{-\mu t_1} - e^{-\mu t_2})^2 \mu^{2\delta} = |e^{-\mu t_1} - e^{-\mu t_2}| \cdot |(e^{-\mu t_1} - e^{-\mu t_2}) \mu^{2\delta}| \leq |(e^{-\mu t_1} - e^{-\mu t_2}) \mu^{2\delta}| \leq |t_1 - t_2| \cdot \max_{\tau \in [t_1, t_2]} \mu^{1+2\delta} e^{-\mu \tau} = |t_1 - t_2| \cdot \mu^{1+2\delta} e^{-\mu t_1}$, $\mu > 0$. Calculations give $\max_{\mu > 0} \mu^{1+2\delta} e^{-\mu t_1} = e^{-(1+2\delta)} \left(\frac{1+2\delta}{t_1}\right)^{1+2\delta}$. Hence, for any $k \in N$, one has

$$(e^{-\lambda_k t_1} - e^{-\lambda_k t_2})^2 \lambda_k^{2\delta} \leq |t_1 - t_2| \cdot e^{-(1+2\delta)} (1+2\delta)^{(1+2\delta)} \cdot (\min\{t_1, t_2\})^{-(1+2\delta)}$$

The last estimate and (22) give

$$\|A^\delta (e^{-At_1} v - e^{-At_2} v)\| \leq D_\delta (\min\{t_1, t_2\})^{-(\delta+\frac{1}{2})} \cdot \sqrt{|t_1 - t_2|} \cdot \|v\|, \quad (23)$$

where $D_\delta \equiv e^{-(\delta+\frac{1}{2})} (\delta + \frac{1}{2})^{(\delta+\frac{1}{2})}$. In the same way we get

$$\|A^\delta (e^{-At_1} v - e^{-At_2} v)\| \leq \widehat{D}_\delta (\min\{t_1, t_2\})^{-(1+\delta)} \cdot |t_1 - t_2| \cdot \|v\|, \quad (24)$$

where $\widehat{D}_\delta \equiv e^{-(1+\delta)} (1+\delta)^{1+\delta}$ and also (see e.g. [Chueshov (1999), (2.8.6)]), we get $\|A^\delta e^{-As}\| \leq (e \cdot s)^{-\delta} \delta^\delta$. Calculations and the last estimate give for $0 < t_1 < t_2$

$$\int_{t_1}^{t_2} \|A^\delta e^{-A(t_2-\tau)}\| d\tau \leq \left(\frac{\delta}{e}\right)^\delta \cdot \frac{|t_1 - t_2|^{1-\delta}}{1-\delta}. \quad (25)$$

Let us consider for any mild solution $u(t)$, the function $G(t) \equiv F(u_t) + du(t)$ and the difference (see (3))

$$\begin{aligned} \|A^\delta(u(t_1) - u(t_2))\| &\leq \|A^\delta(e^{-At_1}\varphi(0) - e^{-At_2}\varphi(0))\| \\ &+ \int_0^{t_1} \|A^\delta(e^{-A(t_1-\tau)}G(\tau) - e^{-A(t_2-\tau)}G(\tau))\| d\tau + \int_{t_1}^{t_2} \|A^\delta e^{-A(t_2-\tau)}G(\tau)\| d\tau. \end{aligned}$$

Here, as before $0 < t_1 < t_2$. The last estimate, (23) and (25) give

$$\|A^\delta(u(t_1) - u(t_2))\| \leq L(\delta, t_1, t_2, \varphi) \cdot \sqrt{|t_1 - t_2|}, \quad (26)$$

where

$$\begin{aligned} L(\delta, t_1, t_2, \varphi) &\equiv D_\delta(\min\{t_1, t_2\})^{-(\delta+\frac{1}{2})} \|\varphi(0)\| \\ &+ \left[D_\delta t_1^{\frac{1}{2}-\delta} \left(\frac{1}{2} - \delta\right)^{-1} + \delta^\delta (e(1-\delta))^{-1} \right] \cdot \max_{\tau \in [0, t_2]} \|F(u_\tau) + du(\tau)\|. \end{aligned} \quad (27)$$

Here we used

$$\begin{aligned} &\int_0^{t_1} \|A^\delta(e^{-A(t_1-\tau)}G(\tau) - e^{-A(t_2-\tau)}G(\tau))\| d\tau \\ &\leq D_\delta \sqrt{|t_1 - t_2|} \max_{\tau \in [0, t_2]} \|G(\tau)\| \cdot \int_0^{t_1} (t_1 - \tau)^{-(\delta+\frac{1}{2})} d\tau \\ &= D_\delta \sqrt{|t_1 - t_2|} \max_{\tau \in [0, t_2]} \|G(\tau)\| \cdot t_1^{\frac{1}{2}-\delta} \left(\frac{1}{2} - \delta\right)^{-1}. \end{aligned}$$

It is evidently that (21) gives $\max_{\tau \in [0, t_2]} \|F(u_\tau) + du(\tau)\| < C$ for any mild solution which is already in the ball of dissipation i.e. (21) holds. These, the form of the constant $L(\delta, t_1, t_2, \varphi)$ (see (27)) and (26) imply that for any time interval $[a, b] \subset (0, +\infty)$ with $a > t(B, \delta)$ (see (21)), there exists a constant $L > 0$ such that for any mild solution $u(t)$ and any $t_1, t_2 \in [a, b]$ one has

$$\|A^\delta(u(t_1) - u(t_2))\| \leq L \cdot \sqrt{|t_1 - t_2|}. \quad (28)$$

This gives the equicontinuity of the family $\{u(t+\theta) \mid \theta \in [-r, 0], t > t(B, \delta)\}$ in all spaces $C_\delta \equiv C([-r, 0]; D(A^\delta))$, $\forall \delta \in [0, \frac{1}{2})$.

Finally, estimate (28) for $\delta \in [0, \frac{1}{2})$ and estimate (21) for $\delta_1 \in (\delta, \frac{1}{2})$, particularly mean (by Arzela-Ascoli theorem) that for any $\varphi \in C$ and $t > t(B, \delta)$ (see (21)), one has $S_t \varphi \in K_\delta$, where K_δ is a compact set in all spaces $C_\delta \equiv C([-r, 0]; D(A^\delta))$, $\forall \delta \in [0, \frac{1}{2})$.

That means that the dynamical system $(S_t; C)$ is dissipative and asymptotically compact, hence by the classical theorem on the existence of an attractor (see, for example, [Babin and Vishik (1992), Temam (1988)]) $(S_t; C)$ has a compact global attractor. The proof of Theorem 2 is complete. ■

Remark 11. *All the results above are valid for local in space variable nonlinearity i.e. equation (1) with*

$$(F_\ell(u_t))(x) \equiv b(u(t - \eta(u_t), x)), \quad x \in \Omega. \quad (29)$$

As an application we can consider the diffusive Nicholson's blowflies equation (see e.g. [So and Yang (1998), So, Wu and Yang (2000)]) with state-dependent delays. More precisely, we consider equation (1) where $-A$ is the Laplace operator with the Dirichlet boundary conditions, $\Omega \subset R^{n_0}$ is a bounded domain with a smooth boundary, the function f can be, for example, $f(s) = \frac{1}{\sqrt{4\pi\alpha}} e^{-s^2/4\alpha}$, as in [So, Wu and Zou (2001)] (for the non-local in space problem) or Dirac delta-function to get the local problem (see Remark 11), the nonlinear function b is given by $b(w) = p \cdot w e^{-w}$. Function b is bounded, so for any continuous delay function η , satisfying (H), the conditions of Theorems 1,2 are valid. As a result, we conclude that the initial value problem (1),(2) is well posed in C and the dynamical system (S_t, C) has a global attractor (Theorem 2).

Remark 12. *All the considerations above are obviously valid for O.D.E.s, for example, of the following form*

$$\dot{u}(t) + Au(t) + d \cdot u(t) = b(u(t - \eta(u_t))), \quad u(\cdot) \in R^n, d \geq 0. \quad (30)$$

One simply needs to substitute $L^2(\Omega)$ by R^n and use $C \equiv C([-r, 0]; R^n)$ instead of $C([-r, 0]; L^2(\Omega))$. Assumptions on the delay function η are the same. The function $b : R^n \rightarrow R^n$ is locally Lipschitz continuous and satisfies $\|b(w)\|_{R^n} \leq C_1 \|w\|_{R^n} + C_b$ with $C_1, C_b \geq 0$. For the method of steps in the case of O.D.E.s see e.g. [Walther (2002), Proposition 1].

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